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RELATIVISTIC CHILD LAW

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The nonrelativistic space-charge theory for a steady one-dimensional current flow across an otherwise evacuated space between plane electrodes leads to a theoretical maximum current density j_m for a given potential difference V_l and electrode separation l determined by the Child law[1].

$$j_m = (-2eV_l^3/m)^{1/2}\Big/(9\pi l^2) \tag{1}$$

Ivey[2] obtained a solution to the corresponding problem of charge carriers accelerated from rest to relativistic speeds in terms of infinite series. However, the relativistic result can be written more concisely in terms of elliptic integrals.

For the relativistic case the Poisson equation

$$d^2V/dx^2 = -4\pi\rho \tag{2}$$

becomes, in dimensionless form,

$$d^2\varphi/d\xi^2 \equiv \varphi'' = (4/9)\sqrt{2\epsilon}\, J(1 + \epsilon\varphi)\left[(1 + \epsilon\varphi)^2 - 1\right]^{-1/2} \tag{3}$$

where, from continuity

$$j = \rho v \tag{4}$$

from conservation of energy,

$$-eV = (\gamma - 1)mc^2 \tag{5}$$

and, by definition,

$$\varphi \equiv V/V_l \tag{6}$$

$$\xi \equiv x/l \tag{7}$$

$$\epsilon \equiv -eV_l/mc^2 \tag{8}$$

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$$J \equiv j/j_m \tag{9}$$

$$\gamma \equiv \left[1 - (v/c)^2\right]^{-1/2} \tag{10}$$

with the electric field $\mathcal{E} = 0$, velocity $v = 0$, and dimensionless potential $\varphi = 0$ at $\xi = 0$. (The quantities ρ, e, v, and V are signed quantities.) Following an elementary integration of Eq. (3) the dimensionless electric field is obtained in the form

$$\varphi' \equiv -l\mathcal{E}/V_l = \left\{(8/9)(2/\epsilon)^{1/2}J\left[(1 + \epsilon\varphi)^2 - 1\right]^{1/2}\right\}^{1/2} \tag{11}$$

Equation (11) cannot be solved in terms of elementary functions; consequently, Ivey obtained a solution using series expansions. In integral form Eq. (11) becomes

$$\xi = (\epsilon/2)^{1/4}(8J/9)^{-1/2}\ g(\varphi;\epsilon) \tag{12}$$

where

$$g(\varphi;\epsilon) \equiv \int_0^{\varphi} d\varphi\left[\epsilon\varphi(2 + \epsilon\varphi)\right]^{-1/4} \tag{13}$$

For $\xi = 1$, Eq. (12) leads to

$$J = (9/8)(\epsilon/2)^{1/2}g^2(\varphi = 1;\epsilon) \tag{14}$$

which determines the ratio between the relativistic current density j and the Child-law current density j_m. Substituting the preceding expression for J into Eq. (12) results in an implicit equation for the dimensionless potential distribution φ, independent of J, that is,

$$\xi = g(\varphi;\epsilon)/g(\varphi = 1;\epsilon) \tag{15}$$

The integral in Eq. (13) possesses a known solution. Let

$$v/c \equiv \tanh\omega \tag{16}$$

Then, considering Eqs. (5), (8), and (10),

$$\omega = \cosh^{-1}(1 + \epsilon\varphi) \tag{17}$$

Equation (13) becomes

$$g(\omega_1;\epsilon) = \epsilon^{-1} \int_0^{\omega_1} (\sinh \omega)^{1/2}\, d\omega \tag{18}$$

where ω_1 corresponds to the upper limit in Eq. (13). Eq. (18) possesses the solution[3]

$$g(\psi;\epsilon) = (2/\epsilon)\Big[\sin \psi(1 - \tfrac{1}{2} \sin^2 \psi)^{1/2}(1 + \cos \psi)^{-1} + (1/2)F(\psi,1/\sqrt{2}) - E(\psi,1/\sqrt{2})\Big] \tag{19}$$

where

$$\psi \equiv \cos^{-1}\Big[(1 - \sinh \omega_1)/(1 + \sinh \omega_1)\Big] \tag{20}$$

and $F(\psi,1/\sqrt{2})$ and $E(\psi,1/\sqrt{2})$ are incomplete elliptic integrals of the first and second kinds, respectively. For values of ψ outside the tabulated range $0 \leq \psi \leq \pi/2$, $F(\psi,1/\sqrt{2})$ and $E(\psi,1/\sqrt{2})$ are, respectively, given by

$$\pm F(\psi,1/\sqrt{2}) = F(n\pi \pm \psi,1/\sqrt{2}) - 2nK \tag{21}$$

and

$$\pm E(\psi,1/\sqrt{2}) = E(n\pi \pm \psi,1/\sqrt{2}) - 2nE \tag{22}$$

where K and E are complete elliptic integrals of the first and second kinds, respectively, and n is an integer.

To complete the description, the charge-density distribution is obtained by substituting the expressions for V and x given by Eqs. (6) and (7), respectively, in Eq. (2), replacing the resulting φ'' by the right side of Eq. (3), and solving for ρ. Then, defining

$$R \equiv \rho/\rho_{ml} \tag{23}$$

where

$$\rho_{ml} = j_m(-2eV_l/m)^{-1/2} = -V_l/(9\pi l^2) \tag{24}$$

is the charge density at $x = l$ in the nonrelativistic limit, and applying Eqs. (8) and (9), the charge density distribution is obtained in the dimensionless form

$$R = \sqrt{2\epsilon}\, J(1 + \epsilon\varphi)\left[\epsilon\varphi(2 + \epsilon\varphi)\right]^{-1/2} \tag{25}$$

The transit time t is in agreement with that derived by Ivey.[2] In a nondimensional form,

$$T = J^{-1/2}\left[(2 + \epsilon)/2\right]^{1/4} \tag{26}$$

where

$$T \equiv t/t_m \tag{27}$$

with t_m the nonrelativistic transit time for a Child-law current.

All of the relativistic formulas reduce, as required, to the corresponding known space-charge formulas in the nonrelativistic limit.

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3. Byrd, P. F. and Friedman, M. D.: Handbook of Elliptic Integrals for Engineers and Physicists (Springer-Verlag, Berlin, 1954), p. 184.